Duality of Decaying Turbulence to a Solvable String Theory

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The oldest unsolved problem in classical physics

- Richard Feynman wrote in 1964 in his famous "Lectures in Physics"
  "there is a physical problem that is common to many fields, that is very old, and that has not been solved. It is not the problem of finding new fundamental particles, but something left over from a long time ago—over a hundred years. Nobody in physics has really been able to analyze it mathematically satisfactorily in spite of its importance to the sister sciences. It is the analysis of circulating or turbulent fluids."
- This problem, in my opinion, **has now been solved**, and today I will present this solution, quite unusual for turbulence experts.
- The **String Theory/QFT** community may find it easier to understand, as it is a manifestation of **duality**, much like **AdS/CFT**.
- For classical physics, this solution is interesting as a new phenomenon of **spontaneous quantization** of a nonlinear dynamical system.

Evolution of Gibbs Distribution

- The Hamiltonian systems satisfy the Liouville equation for the probability density in phase space

$$\partial_t \rho = \{H, \rho\}, \tag{1}$$

- Based on this equation, Theoretical Physics conjectured the **Gibbs distribution**

$$\rho = \exp(-\beta H); \tag{2}$$

as the only conserved multiplicative measure.

- In dissipative systems, such as the NS, the probability measure is not conserved but decays into a standstill with zero velocity.
- So, the turbulence problem is to find **decaying solution for the probability evolution, to replace the Gibbs distribution**.
- This solution must be **degenerate** (fixed manifold rather than a fixed point) to describe critical phenomena observed in turbulent flows.

Loop Average and Dimension Reduction

- The loop average is defined as the **characteristic function** for the distribution of velocity circulation:

$$\Psi[\gamma, C] = \left\langle \exp\left(\frac{\imath\gamma}{\nu}\Gamma_C\right)\right\rangle, \quad \Gamma_C = \oint d\vec{C}(\theta) \cdot \vec{v}(\vec{C}(\theta)).$$

- The loop functional, just like a Wilson loop in Abelian gauge theory, is **invariant under adding a gradient of a scalar field** to the velocity.
- This is a particular case of the **Hopf functional** representation:

$$\Psi[\gamma, C] = \left\langle \exp\left(\int_{\vec{r}\in\mathbb{R}^3} \vec{J}_C(\vec{r}) \cdot \vec{v}(\vec{r})\, d^3 r\right)\right\rangle.$$

- An imaginary source $\vec{J}(\vec{r})$ is concentrated on a fixed loop in space $\mathbb{R}_3$

$$\vec{J}_C(\vec{r}) = \frac{\imath\gamma}{\nu} \oint d\vec{C}(\theta)\delta^3\left(\vec{r} - \vec{C}(\theta)\right)$$

- The evolution **equation for the loop average replaces the Liouville equation** for the dissipative system.

Loop Equation as Quantum Mechanics in Loop Space

- The velocity dynamics is governed by the **incompressible Navier-Stokes equation** (with vorticity $\vec{\omega} = \vec{\nabla} \times \vec{v}$)

$$\partial_t \vec{v} = \vec{v} \times \vec{\omega} - \nu \vec{\nabla} \times \vec{\omega} - \vec{\nabla}\left(p + \frac{\vec{v}^2}{2}\right); \; \vec{\nabla} \cdot \vec{v} = 0; \qquad (3)$$

- The loop functional satisfies a **key relation** derived from this equation (with gradient terms integrating to zero on a closed loop):

$$\imath\nu\partial_t \Psi[\gamma, C] = \left\langle \gamma \oint d\vec{C}(\theta) \cdot \left(\nu \vec{\nabla} \times \vec{\omega} - \vec{v} \times \vec{\omega}\right) e^{\frac{\imath\gamma}{\nu}\Gamma_C} \right\rangle.$$

- This relation leads to the **closed functional equation** for the loop average [6]:

$$\imath\nu\partial_t \Psi[\gamma, C] = \oint d\vec{C}(\theta) \cdot \hat{L}\left[\frac{\delta}{\delta \vec{C}(\cdot)}\right] \Psi[\gamma, C].$$

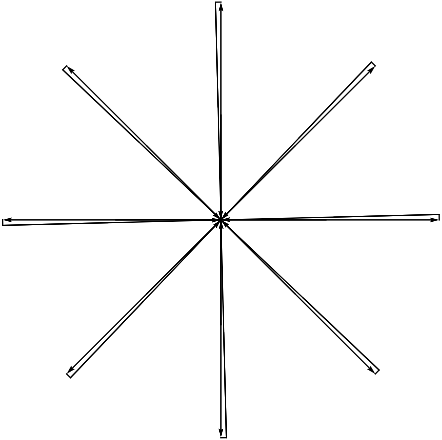
Vorticity correlation functions

- To define and explain the loop operator $\hat{L}$, we need to revive a Loop calculus developed in gauge theories in the 80-ties and 90-ties.
- The area derivatives of the loop functional bring down vorticity $\vec{\omega} = \vec{\nabla} \times \vec{v}$, which is an analog of the field strength in QED.

$$\nu \frac{\delta \Psi[C,t]}{\delta \vec{\sigma}} = \imath \gamma \left\langle \vec{\omega} \exp \frac{\imath \gamma \Gamma_C[v]}{\nu} \right\rangle_{NS} \tag{4}$$

- Applying these area derivatives at n points $\vec{r}_i = \vec{C}(\theta_i)$ we **bring down from exponential n factors of vorticity** $\vec{\omega}(\vec{r}_i)$. After that, we **contract the loop to a set of spokes** $C_S(\vec{r}_1, \dots \vec{r}_n)$, backtracking from the origin to these points (see a figure on the next slide). The **circulation $\Gamma_C[v]$ vanishes** for this loop.

Bike wheel



Vorticity correlation functions

- The positions of the angles $\theta_1 \dots \theta_n$ **do not affect the loop average** as long as these points are ordered on a loop (parametric invariance); so we can integrate them out.
- We arrive at the relation for the **vorticity correlation in the loop dynamics**:

$$\left\langle \vec{\omega}(r_1) \otimes \vec{\omega}(r_2) \cdots \otimes \vec{\omega}(r_n) \right\rangle_v = \frac{n!}{(2\pi)^n}$$

$$\left\langle \int \cdots \int_{0<\theta_1<\dots<\theta_n<2\pi} d\theta_1 \hat{\omega}(\theta_1) \otimes d\theta_2 \hat{\omega}(\theta_2) \cdots \otimes d\theta_n \hat{\omega}(\theta_n) e^{\frac{\imath \Gamma_C}{\nu}} \right\rangle_{C=C_s(\dots)} \quad (5)$$

$$\hat{\omega}(\theta) = -\imath\nu \frac{\delta}{\delta \vec{\sigma}(\theta)} \quad (6)$$

Loop Calculus

- Geometrically, **the area derivative is a vector multiplying the infinitesimal area** $\delta\vec{C}(\theta) \times \vec{C}'$ in a variation of the circulation. By the Stokes theorem,

$$\frac{\delta\Gamma_C}{\delta\vec{C}(\theta)} = \vec{\omega}(\vec{C}(\theta)) \times \vec{C}'(\theta) \tag{7}$$

- The functionals with finite area derivative are called **Stokes functionals**.
- The formal definition of the area derivative is given by the integral **picking the delta function term in the second functional derivative**, as observed by Sasha Polyakov back in the eighties

$$\frac{\delta}{\delta\vec{\sigma}(\theta)} = \frac{\delta}{\delta\vec{C}'(\theta)} \times \int_{\theta-0}^{\theta+0} d\theta' \frac{\delta}{\delta\vec{C}(\theta')} \tag{8}$$

The gradient operator and the Bianchi identity

- The conservation law $\vec{\nabla} \cdot \vec{\omega} = 0$ in coordinate space leads to the **Bianchi identity in the loop space**

$$\hat{D}(\theta) \cdot \frac{\delta \Gamma_C}{\delta \vec{C}(\theta)} = 0 \tag{9}$$

- The gradient operator

$$\hat{D}(\theta) = \int_{\theta-0}^{\theta+0} d\theta' \frac{\delta}{\delta \vec{C}(\theta')} \tag{10}$$

  geometrically means **shifting an infinitesimal vicinity of the point** $\vec{C}(\theta)$ at the curve.
- When applied to any local function of $\vec{C}(\theta)$ like $\vec{\omega}(\vec{C}(\theta))$, this operator is equivalent to a gradient, but **for the circulation it yields zero** in virtue of the Stokes theorem.
- All these operations were **justified mathematically by taking a limit of a polygonal approximation** of the loop.

The loop equation

- Now we can define the loop equation. The operator $\hat{L}$ is obtained from the differential operator in the NS equation for circulation **by replacement** of $\vec{\nabla}, \vec{\omega}$ by above operators

$$\hat{L}(\theta) \exp \frac{\imath\gamma}{\nu}\Gamma_C[v] = \left(-\nu \hat{D}(\theta) \times \hat{\omega}(\theta) + \hat{\omega}(\theta) \times \hat{v}(\theta)\right) \exp \frac{\imath\gamma}{\nu}\Gamma_C[v]; \tag{11}$$

- The operator $\hat{v}(\theta)$ is related to previous operators by the **Biot-Savart formula**

$$\vec{v}(\vec{r}) = \frac{-1}{\vec{\nabla}^2}\vec{\nabla} \times \vec{\omega}(\vec{r}); \tag{12}$$

$$\hat{v}(\theta) = \frac{-1}{\hat{D}(\theta)^2}\hat{\nabla}(\theta) \times \hat{\omega}(\theta); \tag{13}$$

- These formulas were justified by taking a limit of a polygonal approximation of the loop, when these functional derivatives become ordinary gradients.

Key Insight: Plane Wave Solutions in Loop Space

- The loop equation maps fluid dynamics to a **Schrödinger equation in loop space** with a Hamiltonian:

$$\hat{H}_C = \oint d\vec{C}(\theta) \cdot \vec{L}\left[\frac{\delta}{\delta\vec{C}(\cdot)}\right].$$

- A **plane wave solution** emerges naturally, as this Hamiltonian depends only on the canonical momenta but not on the coordinates C

$$\Psi[\gamma, C] = \left\langle \exp\left(\frac{\imath\gamma}{\nu}\oint d\vec{C}(\theta)\cdot\vec{P}(t,\theta)\right)\right\rangle_{P(t)}.$$

- The averaging $\langle \dots \rangle_{P(t)}$ goes over solutions of the following **momentum loop equation (MLE)**, with $\Delta\vec{P} = \vec{P}(\theta_+) - \vec{P}(\theta_-)$ being a discontinuity:

$$\nu\partial_t\vec{P} = -\gamma^2(\Delta\vec{P})^2\vec{P} + \Delta\vec{P}\left(\gamma^2\vec{P}\cdot\Delta\vec{P} + \imath\gamma\left(\frac{(\vec{P}\cdot\Delta\vec{P})^2}{\Delta\vec{P}^2} - \vec{P}^2\right)\right).$$

The plane wave is an eigenvector of the loop operators

- This **dimensional reduction of the fluid dynamics** was hidden in plain sight for centuries!
- The functional derivatives $\frac{\delta}{\delta\vec{C}(\theta)}$ bring down $\vec{P}'(\theta)$ from the exponential; the right side of the equation reduces to a **rational function of the values** $\vec{P}(\theta \pm 0)$:

$$W[C,P] = \exp\left(\frac{\imath\gamma}{\nu}\oint d\vec{C}(\theta)\cdot\vec{P}(t,\theta)\right); \tag{14}$$

$$\hat{D}(\theta)W[C,P] \propto \int_{\theta-0}^{\theta+0} d\theta'\vec{P}'(t,\theta')W[C,P]; \tag{15}$$

$$\hat{\omega}(\theta)W[C,P] \propto \vec{P}(\theta)\times\int_{\theta-0}^{\theta+0} d\theta'\vec{P}'(t,\theta')W[C,P]; \tag{16}$$

$$\int_{\theta-0}^{\theta+0} d\theta'\vec{P}'(t,\theta') = \vec{P}(\theta+0)-\vec{P}(\theta-0) = \Delta\vec{P}(\theta) \tag{17}$$

- The best part is still ahead: the MLE can be **analytically solved!**

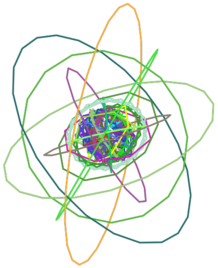
The Euler Ensemble Solves MLE

- The decaying solution of MLE [8] is given by:

$$\vec{P}(\theta, t) = \sqrt{\frac{\nu}{2(t+t_0)}} \frac{\vec{F}(\theta)}{\gamma},$$

  where $\vec{F}(\theta)$ is a **universal fractal curve**, constructed as the limit $N \to \infty$ of a regular star polygon $\{q/p\}$ with vertices: $\vec{F}\left(\frac{2\pi k}{N}\right) = \hat{\Omega} \cdot \frac{\{\cos(\alpha_k), \sin(\alpha_k), \imath \cos(\frac{\beta}{2})\}}{2\sin(\frac{\beta}{2})}$, where: $\beta = \frac{2\pi p}{q}$, $\alpha_k = \beta \sum_{l=0}^{k} \sigma_l$, $k = 1, \dots N, \quad N \to \infty$

- The parameters $\hat{\Omega} \in SO(3), \frac{p}{q} \in \mathbb{Q}, \sigma_k = \pm 1$ are **random**, making $\vec{P}(\theta, t)$ a **fixed stochastic trajectory** of MLE.
- This solution is equivalent to a **random walk on these regular star polygons**.
- **Validation:** This solution has been verified using *Mathematica*® notebooks [3] and rigorously tested in collaboration with mathematicians [2].
- **Significance:** This framework establishes a quantitative link between **classical turbulence** and number theory through a novel mechanism of **spontaneous quantization**.



Dual Amplitude and Loop Functional

- The loop functional in the Euler ensemble corresponds to the **dual amplitude** of string theory, defined on a discrete target space $\vec{F}(\theta)$ with distributed external momentum $\vec{Q}(\theta, t) = \frac{\vec{C}'(\theta)}{\sqrt{2\nu(t+t_0)}}$:

$$\Psi[C, t] = \left\langle \exp \imath \oint d\theta \vec{F}(\theta) \cdot \vec{Q}(\theta, t) \right\rangle_{F,\sigma}$$

- Averaging over **string target space** $\vec{F}_k$ corresponds to summing over star polygons with unit sides and rational angles $\beta = 2\pi \frac{p}{q}$.
- Averaging over **fermionic/Ising degrees of freedom** produces a **random walk** (Brownian motion in the continuum limit) across polygon edges.
- The viscosity enters this string theory as a coupling constant in the denominator of the effective Action. The turbulent limit of $\nu \to 0$ becomes the **weak coupling limit**, solvable in the WKB approximation.

Explicit Formula for Velocity Correlation

The statistical limit of the Euler ensemble as $N \to \infty, \nu \to 0, \tilde{\nu} = \nu N^2 =$ *const* , enables computations of the energy spectrum and correlation functions in quadrature.

Here is the resulting **formula for the second moment of velocity difference**:

$$\left\langle \Delta \vec{v}^2 \right\rangle (r,t) = \frac{\tilde{\nu}^2}{\nu t} \int_{\epsilon - \imath\infty}^{\epsilon + \imath\infty} \frac{dp}{2\pi\imath} V(p) \left(\frac{|\vec{r}|}{\sqrt{\tilde{\nu} t}} \right)^p ;$$
$$V(p) = -\frac{f(-1-p)\zeta\left(\frac{13}{2} - p\right)\csc\left(\frac{\pi p}{2}\right)}{16\pi^2 (p+1)(2p-15)(2p-5)\zeta\left(\frac{15}{2} - p\right)}. \quad (18)$$

Here $f(z)$ is an entire function computed via Mellin integrals of elementary functions. $V(p)$ is meromorphic. ν is physical viscosity, while turbulent viscosity $\tilde{\nu}$ is a free parameter of the solution.

Spectrum of indices of velocity correlation

The spectrum of indices for velocity correlation is given by the poles of $V(p)$

indexes of velocity correlation

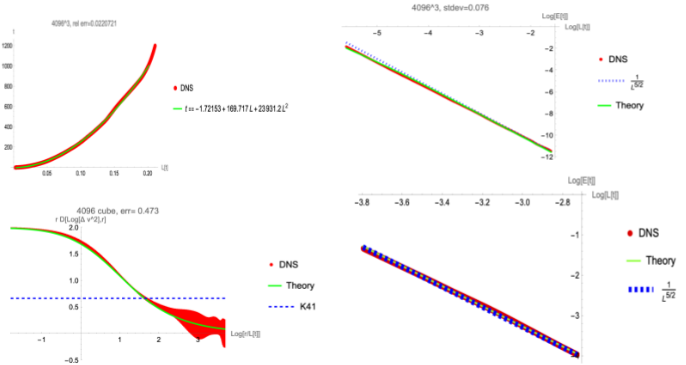
Index	Condition
-1	
0	
$2n$	$n \in \mathbb{Z}, n \geq 1$
$5/2$	
$11/2$	
$\frac{15+4n}{2}$	$n \in \mathbb{Z}, n \geq 0$
$7 \pm \imath t_n$	$n \in \mathbb{Z}$

(19)

where $\frac{1}{2} \pm \imath t_n$ are the **zeros of the Riemann ζ function**.

DNS Data and Theoretical Match

Verification by DNS (Sreenivasan et. al., 2025)



Decaying Energy Multi Scaling laws

$$\Delta_p$$

Leading → $-\frac{5}{4}$

$$-\frac{11}{4}$$

$$-\frac{7}{2} \pm \frac{\imath}{2} t_n \text{ if } n \in \mathbb{Z}$$

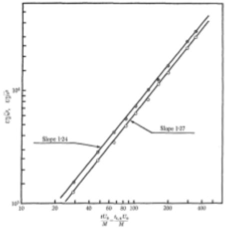
$$-\frac{15}{4} - n \text{ if } n \in \mathbb{Z} \wedge n \geq 0$$

$$\frac{n}{2} \text{ if } n \in \mathbb{Z} \wedge n \geq 0$$

$$\zeta\left(\frac{1}{2} + \imath t_n\right) = 0$$

$$E(t) \propto \sum \Re A_p t^{\Delta_p}$$

Kolmogorov-Saffman model: 6/5 = 1.2



Comte-Bellot G, Corrsin S. data: 1.25

The Nonabelian turbulence

- The MLE can be generalized to the **nonabelian gauge theory** $\hat{A}(\vec{r})$ in the Lie algebra of some semisimple group G (with $\vec{r} \in \mathbb{E}_d$).

$$\partial_t \hat{A}_\beta = \nu \left[\hat{D}_\alpha, \hat{F}_{\alpha\beta}\right]; \tag{20}$$

$$\hat{F}_{\alpha\beta} = \left[\hat{D}_\alpha, \hat{D}_\beta\right]; \tag{21}$$

$$\hat{D}_\alpha = \nabla_\alpha + \hat{A}_\alpha; \tag{22}$$

$$W[C,t] = \left\langle \text{tr}\, \hat{P} \exp \oint \hat{A}_\beta dC_\beta \right\rangle_{A(t)}; \tag{23}$$

$$W[C,0] = \int [\delta A] \text{tr}\, \hat{P} \exp \oint \hat{A}_\beta dC_\beta \exp -\frac{\beta}{2} \int \text{tr}\, \hat{F}^2_{\alpha\beta} \tag{24}$$

- The same Anzatz with **abelian** momentum loop solves this equation

$$W[C,t] = \left\langle \exp \frac{\imath}{\nu} \oint dC_\alpha(\theta) P_\alpha(t,\theta) \right\rangle_{P(t)} \tag{25}$$

The same Euler ensemble

- The new MLE has **only a diffusion term**

$$\nu \partial_t P_\beta = -(\Delta P)^2 P_\beta + \Delta P_\beta P_\alpha \Delta P_\alpha \tag{26}$$

- The initial data distribution $\rho[P]$ corresponds to the path integral of $W[C,0]$

$$\rho[P]_{t=0} \propto \int [\delta C] W[C,0] \exp -\frac{\imath}{\nu} \oint dC_\alpha(\theta) P_\alpha(t,\theta) \tag{27}$$

  and thus it depends on the Lie algebra; otherwise, the **equation is completely universal**.

- The **shifted Euler ensemble** serves as its asymptotic solution corresponding to decaying turbulence

$$\vec{P}(\theta,t) = \sqrt{\frac{\nu}{2(t+t_0)}} \vec{F}(\theta); \tag{28}$$

$$\vec{F}\left(\frac{2\pi k}{N}\right) = \hat{\Omega} \cdot \frac{\{\cos(\alpha_k), \sin(\alpha_k), 0, \dots, 0\}}{2\sin(\frac{\beta}{2})}; \; \Omega \in SO(d) \tag{29}$$

The Future Directions and Remaining Problems

- The **equivalence between decaying turbulence and solvable string theory** offers a **novel perspective** in fluid mechanics.
- This framework provides tools for analyzing:
  - **Turbulence in $d > 3$ dimensions** (solved)[8].
  - **Magnetohydrodynamic (MHD) turbulence** (solved)[4].
  - **Turbulent mixing (passive scalar)** (solved)[5].
  - **Turbulence forced by random rotations** (solved, in preparation)[1].
  - **Compressible (aerodynamic) turbulence**.
- **Collaboration** with mathematicians, experimentalists, and DNS researchers is crucial for **extending and validating** this theory.
- **Open Questions:**
  - **What role does external forcing play** in modifying the turbulence-string duality?
  - Are there **other PDEs** with similar dimensional reductions? (the above nonabelian diffusion (20), maybe more).
  - How can the **Euler ensemble** be generalized for random walks on loop groups?

References I

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[4] Alexander Migdal. *Dual Theory of MHD Turbulence*. https://arxiv.org/abs/2503.12682. 2025. arXiv: 2503.12682 [physics.flu-dyn].

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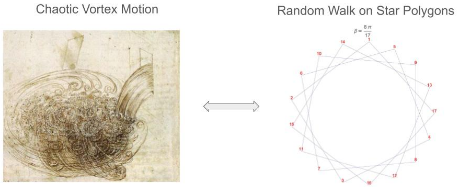
[8] Alexander Migdal. "To the Theory of Decaying Turbulence". In: *Fractal and Fractional* 7.10 (Oct. 2023), p. 754. ISSN: 2504-3110. DOI: 10.3390/fractalfract7100754. arXiv: 2304.13719 [physics.flu-dyn]. URL: http://dx.doi.org/10.3390/fractalfract7100754.

Encore?

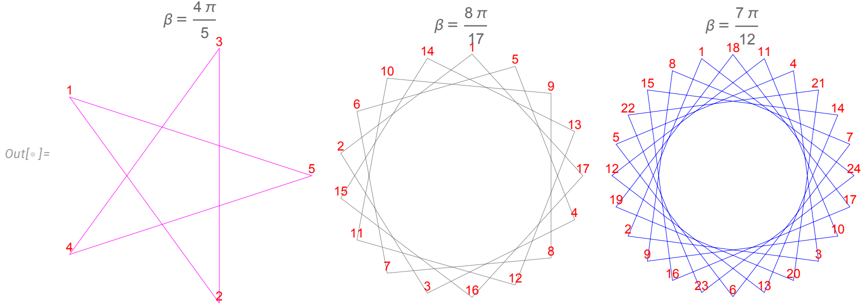
Some afterthoughts, just in case the audience asks for more...

Spontaneous Quantization of Classical Turbulence



- A **thirty-year effort** culminated in 2023 in an **exact, universal solution** to the Navier-Stokes equations in the turbulent regime [8].
- This solution reveals a **duality** between decaying turbulence and a solvable string theory with a **discrete target space**: random walk on regular star polygons.
- Turbulent randomness arises from **spontaneous quantization**, where discrete parameters emerge from a **manifold of solutions of MLE and periodicity requirement**.
- **Universal number-theoretic functions** now quantitatively explain DNS data violations of classical scaling laws, providing a **predictive framework** for turbulence [7].

Regular Star Polygons of the Euler Ensemble



Description

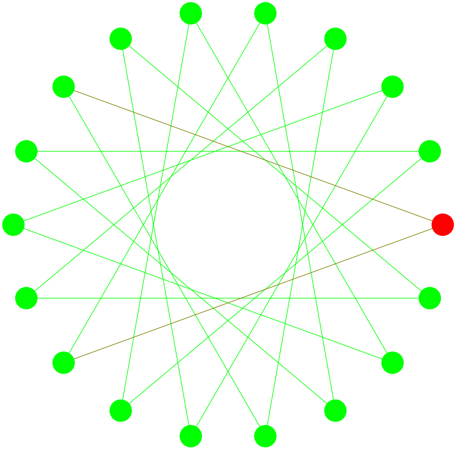
Regular star polygons with various p, q.

These were first classified by **Thomas Bradwardine** (c. 1300 – 1349), Archbishop of Canterbury.

The σ_k **variable** governs the direction of the random step along the link $k \leftrightarrow k+1$.

The **random walk** can traverse the polygon multiple times, provided it returns to its starting point.

Random Walk on Star Polygon, $p = 7, q = 18, N = 100$



Visualization

Visualization of a **random walk** on a star polygon with parameters $p = 7$, $q = 18$, and $N = 100$.

The Quantum Ergodic Hypothesis

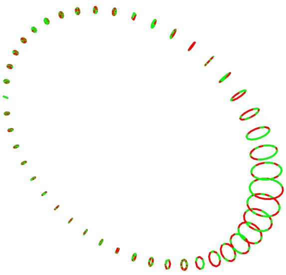
- The complex wave function Ψ of quantum mechanics in loop space equals the characteristic function of a probability distribution P. This relation is exact, and it differs from the conventional $P = |\Psi|^2$.
- Each **distinct state** in a quantum system contributes to the partition function with unit weight, a principle adopted in our **quantum description** of the nonlinear NS system.
- This **uniform measure** is an **additional conjecture**, comparable to the ergodic hypothesis in Newtonian mechanics. While well-established in physics, such hypotheses are mathematically proven only for specific cases.
- The heuristic argument for our quantum ergodic hypothesis is the equivalence of the Euler ensemble to the string theory with discrete target space.

Turbulence/String Duality: Insights

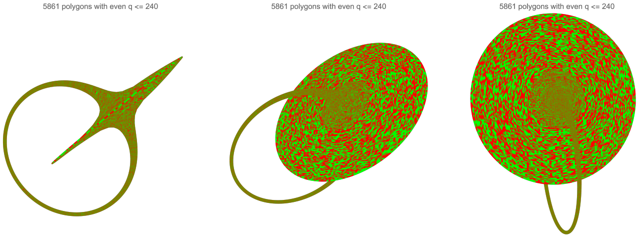
- **Duality phenomena** in physics connect strong coupling in one theory to weak coupling in another, as seen in the **AdS/CFT correspondence**.
- Duality links **statistical averages** between two systems without requiring a direct mapping of their dynamical variables.
- In turbulence, the **target space** of the dual string theory is discrete, represented by **regular star polygons** with unit sides and rational angles, $\beta = 2\pi\frac{p}{q}$.
- Fermionic or Ising degrees of freedom $(\nu_k = 0, 1;\ \sigma_k = \pm 1)$ describe the random walk along polygon edges.
- The radii of these polygons, $R = \frac{1}{2\sin\pi\frac{p}{q}}$, follow number-theoretic distributions involving Euler totients and the Riemann ζ function [7].
- This isn't a conventional string theory with continuous target space —this string exists in three or higher dimensions.

Discrete Symmetry and Target Space Integration

- **Key contrast:** Turbulence features chaotic velocity fields mapping $\mathbb{R}_3 \mapsto \mathbb{R}_3$, whereas the dual string theory has discrete variables $\vec{F}, \sigma, w$ and one continuous variable $\Omega \in SO(3)$ mapping $(\mathbb{Q} \uplus \mathbb{Z}_2 \uplus \mathbb{Z} \uplus SO(3)) \mapsto \mathbb{R}_3$.
- Integration over the target space (star polygons) reduces to a **discrete sum/integral** over Euler ensemble states:
  - Rational numbers $\frac{p}{q} \in \mathbb{Q}$,
  - Ising variables $\sigma_k \in \mathbb{Z}_2$,
  - Winding number $w = \frac{p}{q} \sum \sigma_k \in \mathbb{Z}$.
  - The rotation matrix $\Omega \in SO(3)$.
- Visualizing the polygons for fixed N, ordered by angle β, as a torus in 3D space reveals the **world sheet** of a discrete string. Each polygon forms a cross-section of the torus:
  - Smaller cross-sections correspond to small p, q.
  - Larger cross-sections arise as $q \to \infty$, with p or $q - p$ held fixed.
- Red/green edge coloring indicates random walk directions σ_k.



Revealing a Hidden Identity



- The WKB solution of this theory amounts to finding a classical trajectory for a random walk around the circle and a functional determinant for harmonic fluctuations, averaged over the number theory distribution of rational numbers.
- We are not merely computing turbulence statistics but **unveiling its hidden second identity as a discrete string theory**.
- This theory challenges traditional views developed over the past eighty years, but so far it has survived a rigorous scrutiny.
- Existing experimental and DNS results show **encouraging agreement** with the predictions of the Euler ensemble.

Closing Thoughts: The Ascent to Understanding

- Scaling the summit of turbulence theory is like climbing up the Matterhorn.
- Each milestone—loop equations, dimension reduction, and Euler ensemble—brings us closer to a breathtaking view at the summit.
- The wing-suit figure is me: flying or falling? The landing will tell...